

Kinks and Particles in Non-integrable Quantum Field Theories

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Abstract

In this talk we discuss an elementary derivation of the semi-classical spectrum of neutral particles in two field theories with kink excitations. We also show that, in the non-integrable cases, each vacuum state cannot generically support more than two stable particles, since all other neutral excitations are resonances, which will eventually decay.

1 Introduction

Two-dimensional massive Integrable Quantum Field Theories (IQFTs) have proven to be one of the most successful topics of relativistic field theory, with a large variety of applications to statistical mechanical models. The main reason for this success consists of their simplified on-shell dynamics which is encoded into a set of elastic and factorized scattering amplitudes of their massive particles [1, 2]. The two-particle S-matrix has a very simple analytic structure, with only poles in the physical strip, and it can be computed combining the standard requirements of unitarity, crossing and factorization together with specific symmetry properties of the theory. The complete mass spectrum is obtained looking at the pole singularities of the S-matrix elements. Off-mass shell quantities, such as the correlation functions, can be also determined once the elastic S-matrix and the mass spectrum are known. In fact, one can compute the exact matrix elements of the (semi)local fields on the asymptotic states with the Form Factor (FF) approach [3], and use them to write down the spectral representation of the correlators. By following this approach, it has been possible, for instance, to tackle successfully the long-standing problem of spelling out the mass spectrum and the correlation functions of the two dimensional Ising model in a magnetic field [2, 4], as well as many other interesting problems of statistical physics (for a partial list of them see, for instance, [5]).

The S-matrix approach can be also constructed for massless IQFTs [6, 7, 8, 9], despite the subtleties in defining a scattering theory between massless particles in $(1 + 1)$ dimensions, and turns out to be useful mainly when conformal symmetry is not present. In this case, massless IQFTs generically describe the Renormalization Group trajectories connecting two different Conformal Field Theories, which respectively rule the ultraviolet and infrared limits of all physical quantities along the flows.

Given the large number of remarkable results obtained by the study of IQFTs, one of the most interesting challenges is to extend the analysis to the non-integrable field theories, at least to those obtained as deformations of the integrable ones and to develop the corresponding perturbation theory. The breaking of integrability is expected to considerably increase the difficulties of the mathematical analysis, since scattering processes are no longer elastic. Non-integrable field theories are in fact generally characterized by particle production amplitudes, resonance states and, correspondingly, decay events. All these features strongly effect the analytic structure of the scattering amplitudes, introducing a rich pattern of branch cut singularities, in addition to the pole structure associated to bound and resonance states. For massive non-integrable field theories, a convenient perturbative scheme

was originally proposed in [10] and called Form Factor Perturbation Theory (FFPT), since it is based on the knowledge of the exact Form Factors (FFs) of the original integrable theory. It was shown that, even using just the first order correction of the FFPT, a great deal of information can be obtained, such as the evolution of their particle content, the variation of their masses and the change of the ground state energy. Whenever possible, universal ratios were computed and successfully compared with their value obtained by other means. Recently, for instance, it has been obtained the universal ratios relative to the decay of the particles with higher masses in the Ising model in a magnetic field, once the temperature is displayed away from the critical value [11] (see also the contribution by G. Delfino in this proceedings [12]). For other and important aspects of the Ising model along non-integrable lines see the references [13, 14, 15, 16]. Applied to the double Sine–Gordon model [17], the FFPT has been useful in clarifying the rich dynamics of this non-integrable model. In particular, in relating the confinement of the kinks in the deformed theory to the non-locality properties of the perturbed operator and predicting the existence of a Ising-like phase transition for particular ratios of the two frequencies – results which were later confirmed by a numerical study [18]. The FFPT has been also used to study the spectrum of the $O(3)$ non-linear sigma model with a topological θ term, by varying θ [19, 20].

In this talk I would like to focus the attention on a different approach to tackle some interesting non-integrable models, i.e. those two dimensional field theories with kink topological excitations. Such theories are described by a scalar real field $\varphi(x)$, with a Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - U(\varphi) \ , \quad (1.1)$$

where the potential $U(\varphi)$ possesses several degenerate minima at $\varphi_a^{(0)}$ ($a = 1, 2, \dots, n$), as the one shown in Figure 1. These minima correspond to the different vacua $|a\rangle$ of the associate quantum field theory.

The basic excitations of this kind of models are kinks and anti-kinks, i.e. topological configurations which interpolate between two neighbouring vacua. Semiclassically they correspond to the static solutions of the equation of motion, i.e.

$$\partial_x^2 \varphi(x) = U'[\varphi(x)] \ , \quad (1.2)$$

with boundary conditions $\varphi(-\infty) = \varphi_a^{(0)}$ and $\varphi(+\infty) = \varphi_b^{(0)}$, where $b = a \pm 1$. Denoting by $\varphi_{ab}(x)$ the solutions of this equation, their classical energy density is

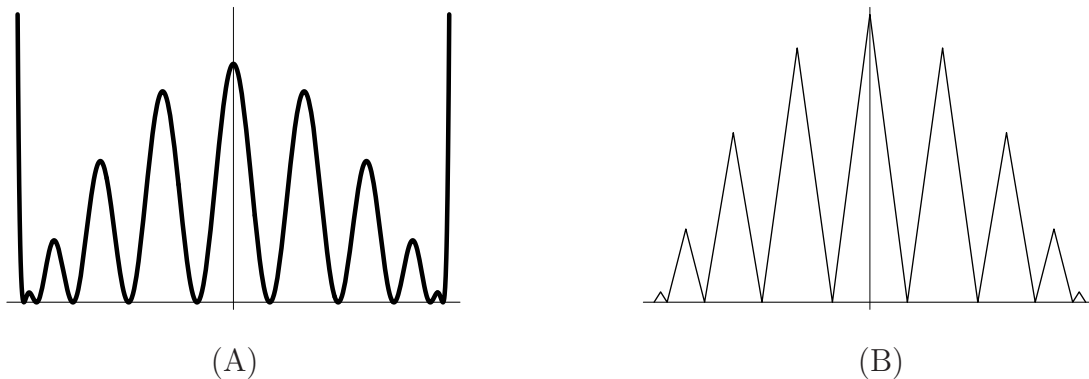


Figure 1: *Potential $U(\varphi)$ of a quantum field theory with kink excitations (A) and histogram of the masses of the kinks (B).*

given by

$$\epsilon_{ab}(x) = \frac{1}{2} \left(\frac{d\varphi_{ab}}{dx} \right)^2 + U(\varphi_{ab}(x)) , \quad (1.3)$$

and its integral provides the classical expression of the kink masses

$$M_{ab} = \int_{-\infty}^{\infty} \epsilon_{ab}(x) . \quad (1.4)$$

It is easy to show that the classical masses of the kinks $\varphi_{ab}(x)$ are simply proportional to the heights of the potential between the two minima $\varphi_a^{(0)}$ and $\varphi_b^{(0)}$: their histogram provides a caricature of the original potential (see Figure 1).

The classical solutions can be set in motion by a Lorentz transformation, i.e. $\varphi_{ab}(x) \rightarrow \varphi_{ab}[(x \pm vt)/\sqrt{1-v^2}]$. In the quantum theory, these configurations describe the kink states $|K_{ab}(\theta)\rangle$, where a and b are the indices of the initial and final vacuum, respectively. The quantity θ is the rapidity variable which parameterises the relativistic dispersion relation of these excitations, i.e.

$$E = M_{ab} \cosh \theta , \quad P = M_{ab} \sinh \theta . \quad (1.5)$$

Conventionally $|K_{a,a+1}(\theta)\rangle$ denotes the *kink* between the pair of vacua $\{|a\rangle, |a+1\rangle\}$ while $|K_{a+1,a}\rangle$ is the corresponding *anti-kink*. For the kink configurations it may be useful to adopt the simplified graphical form shown in Figure 2.

The multi-particle states are given by a string of these excitations, with the adjacency condition of the consecutive indices for the continuity of the field configuration

$$|K_{a_1,a_2}(\theta_1) K_{a_2,a_3}(\theta_2) K_{a_3,a_4}(\theta_3) \dots\rangle , \quad (a_{i+1} = a_i \pm 1) \quad (1.6)$$

In addition to the kinks, in the quantum theory there may exist other excitations in the guise of ordinary scalar particles (breathers). These are the neutral excitations

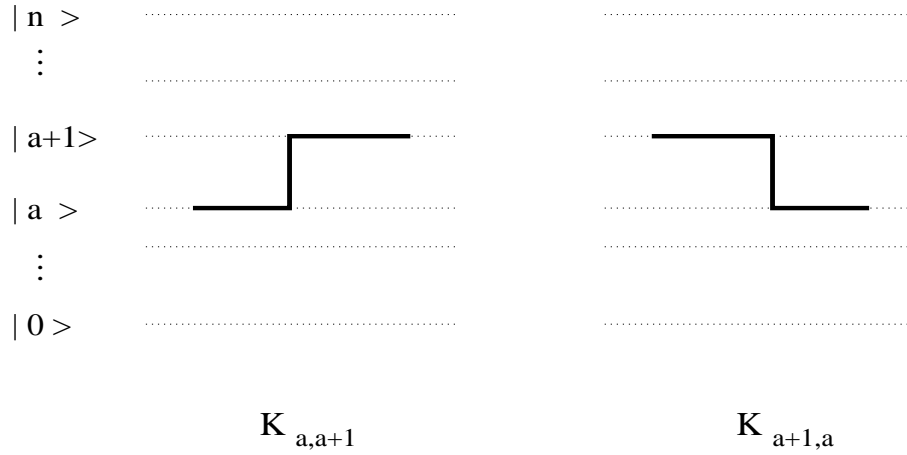


Figure 2: *Kink and antikink configurations.*

$|B_c(\theta)\rangle_a$ ($c = 1, 2, \dots$) around each of the vacua $|a\rangle$. For a theory based on a Lagrangian of a single real field, these states are all non-degenerate: in fact, there are no extra quantities which commute with the Hamiltonian and that can give rise to a multiplicity of them. The only exact (alias, unbroken) symmetries for a Lagrangian as (1.1) may be the discrete ones, like the parity transformation P , for instance, or the charge conjugation \mathcal{C} . However, since they are neutral excitations, they will be either even or odd eigenvectors of \mathcal{C} .

The neutral particles must be identified as the bound states of the kink-antikink configurations that start and end at the same vacuum $|a\rangle$, i.e. $|K_{ab}(\theta_1) K_{ba}(\theta_2)\rangle$, with the “tooth” shapes shown in Figure 3.

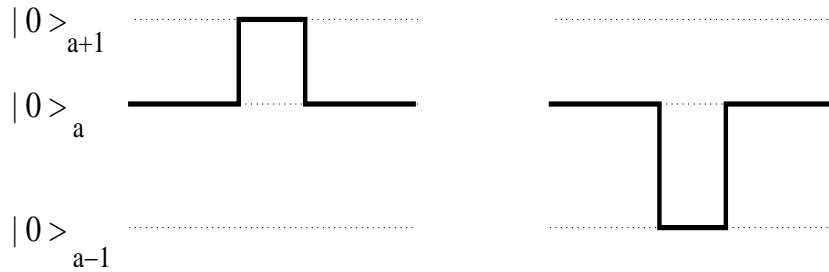


Figure 3: *Kink-antikink configurations which may give rise to a bound state nearby the vacuum $|0\rangle_a$.*

If such two-kink states have a pole at an imaginary value $i u_{ab}^c$ within the physical strip $0 < \text{Im } \theta < \pi$ of their rapidity difference $\theta = \theta_1 - \theta_2$, then their bound states

are defined through the factorization formula which holds in the vicinity of this singularity

$$| K_{ab}(\theta_1) K_{ba}(\theta_2) \rangle \simeq i \frac{g_{ab}^c}{\theta - i u_{ab}^c} | B_c \rangle_a . \quad (1.7)$$

In this expression g_{ab}^c is the on-shell 3-particle coupling between the kinks and the neutral particle. Moreover, the mass of the bound states is simply obtained by substituting the resonance value $i u_{ab}^c$ within the expression of the Mandelstam variable s of the two-kink channel

$$s = 4M_{ab}^2 \cosh^2 \frac{\theta}{2} \longrightarrow m_c = 2M_{ab} \cos \frac{u_{ab}^c}{2} . \quad (1.8)$$

Concerning the vacua themselves, as well known, in the infinite volume their classical degeneracy is removed by selecting one of them, say $| k \rangle$, out of the n available. This happens through the usual spontaneously symmetry breaking mechanism, even though – strictly speaking – there may be no internal symmetry to break at all. This is the case, for instance, of the potential shown in Figure 1, which does not have any particular invariance. In the absence of a symmetry which connects the various vacua, the world – as seen by each of them – may appear very different: they can have, indeed, different particle contents. The problem we would like to examine in this talk concerns the neutral excitations around each vacuum, in particular the question of the existence of such particles and of the value of their masses. To this aim, let's make use of a semiclassical approach.

2 A semiclassical formula

The starting point of our analysis is a remarkably simple formula due to Goldstone-Jackiw [23], which is valid in the semiclassical approximation, i.e. when the coupling constant goes to zero and the mass of the kinks becomes correspondingly very large with respect to any other mass scale. In its refined version, given in [24] and rediscovered in [25], it reads as follows¹ (Figure 4)

$$f_{ab}^\varphi(\theta) = \langle K_{ab}(\theta_1) | \varphi(0) | K_{ab}(\theta_2) \rangle \simeq \int_{-\infty}^{\infty} dx e^{iM_{ab}\theta x} \varphi_{ab}(x) , \quad (2.9)$$

where $\theta = \theta_1 - \theta_2$.

¹The matrix element of the field $\varphi(y)$ is easily obtained by using $\varphi(y) = e^{-iP_\mu y^\mu} \varphi(0) e^{iP_\mu y^\mu}$ and by acting with the conserved energy-momentum operator P_μ on the kink state. Moreover, for the semiclassical matrix element $F_{ab}^G(\theta)$ of the operator $G[\varphi(0)]$, one should employ $G[\varphi_{ab}(x)]$. For instance, the matrix element of $\varphi^2(0)$ are given by the Fourier transform of $\varphi_{ab}^2(x)$.

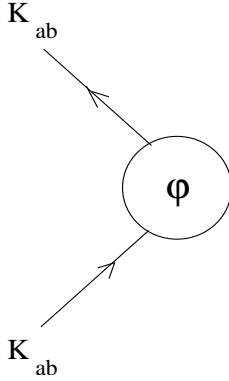


Figure 4: *Matrix element between kink states.*

Notice that, if we substitute in the above formula $\theta \rightarrow i\pi - \theta$, the corresponding expression may be interpreted as the following Form Factor

$$F_{ab}^{\varphi}(\theta) = f(i\pi - \theta) = \langle a \mid \varphi(0) \mid K_{ab}(\theta_1) K_{ba}(\theta_2) \rangle . \quad (2.10)$$

In this matrix element, it appears the neutral kink states around the vacuum $\mid a \rangle$ we are interested in.

Eq. (2.9) deserves several comments.

1. The appealing aspect of the formula (2.9) stays in the relation between the Fourier transform of the *classical* configuration of the kink, – i.e. the solution $\varphi_{ab}(x)$ of the differential equation (1.2) – to the *quantum* matrix element of the field $\varphi(0)$ between the vacuum $\mid a \rangle$ and the 2-particle kink state $\mid K_{ab}(\theta_1) K_{ba}(\theta_2) \rangle$.

Once the solution of eq. (1.2) has been found and its Fourier transform has been taken, the poles of $F_{ab}(\theta)$ within the physical strip of θ identify the neutral bound states which couple to φ . The mass of the neutral particles can be extracted by using eq. (1.8), while the on-shell 3-particle coupling g_{ab}^c can be obtained from the residue at these poles (Figura 5)

$$\lim_{\theta \rightarrow iu_{ab}^c} (\theta - iu_{ab}^c) F_{ab}(\theta) = i g_{ab}^c \langle a \mid \varphi(0) \mid B_c \rangle . \quad (2.11)$$

2. It is important to stress that, for a generic theory, the classical kink configuration $\varphi_{ab}(x)$ is not related in a simple way to the anti-kink configuration $\varphi_{ba}(x)$. It is precisely for this reason that neighbouring vacua may have a different spectrum of neutral excitations, as shown in the examples discussed in the following sections.

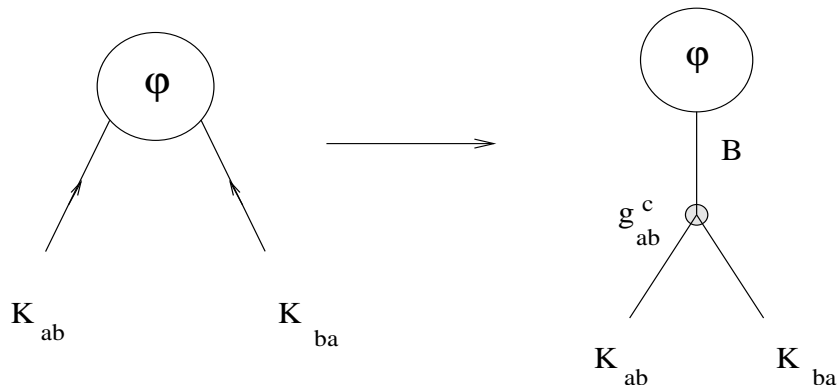


Figure 5: *Residue equation for the matrix element on the kink states.*

3. It is also worth noting that this procedure for extracting the bound states masses permits in many cases to avoid the semiclassical quantization of the breather solutions [22], making their derivation much simpler. The reason is that, the classical breather configurations depend also on time and have, in general, a more complicated structure than the kink ones. Yet, it can be shown that in non-integrable theories these configurations do not exist as exact solutions of the partial differential equations of the field theory. On the contrary, in order to apply eq. (2.9), one simply needs the solution of the *ordinary* differential equation (1.2). It is worth notice that, to locate the poles of $f_{ab}^\varphi(\theta)$, one only needs to looking at the exponential behavior of the classical solutions at $x \rightarrow \pm\infty$, as discussed below.

In the next two sections we will present the analyse a class of theories with only two vacua, which can be either symmetric or asymmetric ones. A complete analysis of other potentials can be found in the original paper [27].

3 Symmetric wells

A prototype example of a potential with two symmetric wells is the φ^4 theory in its broken phase. The potential is given in this case by

$$U(\varphi) = \frac{\lambda}{4} \left(\varphi^2 - \frac{m^2}{\lambda} \right)^2. \quad (3.12)$$

Let us denote with $|\pm 1\rangle$ the vacua corresponding to the classical minima $\varphi_{\pm}^{(0)} = \pm \frac{m}{\sqrt{\lambda}}$. By expanding around them, $\varphi = \varphi_{\pm}^{(0)} + \eta$, we have

$$U(\varphi_{\pm}^{(0)} + \eta) = m^2 \eta^2 \pm m\sqrt{\lambda} \eta^3 + \frac{\lambda}{4} \eta^4 . \quad (3.13)$$

Hence, perturbation theory predicts the existence of a neutral particle for each of the two vacua, with a bare mass given by $m_b = \sqrt{2}m$, irrespectively of the value of the coupling λ . Let's see, instead, what is the result of the semiclassical analysis.

The kink solutions are given in this case by

$$\varphi_{-a,a}(x) = a \frac{m}{\sqrt{\lambda}} \tanh \left[\frac{mx}{\sqrt{2}} \right] , \quad a = \pm 1 \quad (3.14)$$

and their classical mass is

$$M_0 = \int_{-\infty}^{\infty} \epsilon(x) dx = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} . \quad (3.15)$$

The value of the potential at the origin, which gives the height of the barrier between the two vacua, can be expressed as

$$U(0) = \frac{3m}{8\sqrt{2}} M_0 , \quad (3.16)$$

and, as noticed in the introduction, is proportional to the classical mass of the kink.

If we take into account the contribution of the small oscillations around the classical static configurations, the kink mass gets corrected as [22]

$$M = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} - m \left(\frac{3}{\pi\sqrt{2}} - \frac{1}{2\sqrt{6}} \right) + \mathcal{O}(\lambda) . \quad (3.17)$$

It is convenient to define

$$c = \left(\frac{3}{2\pi} - \frac{1}{4\sqrt{3}} \right) > 0 ,$$

and also the adimensional quantities

$$g = \frac{3\lambda}{2\pi m^2} \quad ; \quad \xi = \frac{g}{1 - \pi c g} . \quad (3.18)$$

In terms of them, the mass of the kink can be expressed as

$$M = \frac{\sqrt{2}m}{\pi \xi} = \frac{m_b}{\pi \xi} . \quad (3.19)$$

Since the kink and the anti-kink solutions are equal functions (up to a sign), their Fourier transforms have the same poles. Hence, the spectrum of the neutral particles

will be the same on both vacua, in agreement with the Z_2 symmetry of the model. We have

$$f_{-a,a}(\theta) = \int_{-\infty}^{\infty} dx e^{iM\theta x} \varphi_{-a,a}(x) = i a \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh\left(\frac{\pi M}{\sqrt{2}m}\theta\right)} .$$

By making now the analitical continuation $\theta \rightarrow i\pi - \theta$ and using the above definitions (3.18), we arrive to

$$F_{-a,a}(\theta) = \langle a \mid \varphi(0) \mid K_{-a,a}(\theta_1) K_{a,-a}(\theta_2) \rangle \propto \frac{1}{\sinh\left(\frac{(i\pi-\theta)}{\xi}\right)} . \quad (3.20)$$

The poles of the above expression are located at

$$\theta_n = i\pi (1 - \xi n) \quad , \quad n = 0, \pm 1, \pm 2, \dots \quad (3.21)$$

and, if

$$\xi \geq 1 \quad , \quad (3.22)$$

none of them is in the physical strip $0 < \text{Im } \theta < \pi$. Consequently, in the range of the coupling constant

$$\frac{\lambda}{m^2} \geq \frac{2\pi}{3} \frac{1}{1 + \pi c} = 1.02338\dots \quad (3.23)$$

the theory does not have any neutral bound states, neither on the vacuum to the right nor on the one to the left. Viceversa, if $\xi < 1$, there are $n = \left[\frac{1}{\xi}\right]$ neutral bound states, where $[x]$ denote the integer part of the number x . Their semiclassical masses are given by

$$m_b^{(n)} = 2M \sin\left[n \frac{\pi \xi}{2}\right] = n m_b \left[1 - \frac{3}{32} \frac{\lambda^2}{m^4} n^2 + \dots\right] . \quad (3.24)$$

Note that the leading term is given by multiples of the mass of the elementary boson $|B_1\rangle$. Therefore the n -th breather may be considered as a loosely bound state of n of it, with the binding energy provided by the remaining terms of the above expansion. But, for the non-integrability of the theory, all particles with mass $m_n > 2m_1$ will eventually decay. It is easy to see that, if there are at most two particles in the spectrum, it is always valid the inequality $m_2 < 2m_1$. However, if $\xi < \frac{1}{3}$, for the higher particles one always has

$$m_k > 2m_1 \quad , \quad \text{for } k = 3, 4, \dots n \quad . \quad (3.25)$$

According to the semiclassical analysis, the spectrum of neutral particles of φ^4 theory is then as follows: (i) if $\xi > 1$, there are no neutral particles; (ii) if $\frac{1}{2} < \xi < 1$, there

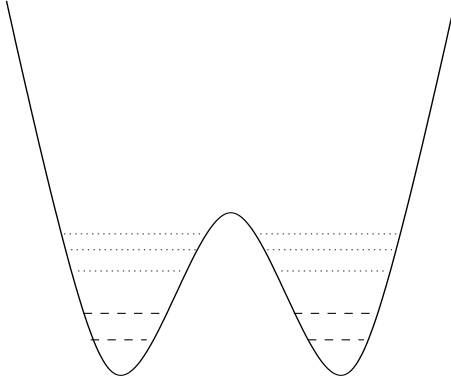


Figure 6: *Neutral bound states of φ^4 theory for $g < 1$. The lowest two lines are the stable particles whereas the higher lines are the resonances.*

is one particle; (iii) if $\frac{1}{3} < \xi < \frac{1}{2}$ there are two particles; (iv) if $\xi < \frac{1}{3}$ there are $\left[\frac{1}{\xi}\right]$ particles, although only the first two are stable, because the others are resonances.

Let us now briefly mention some general features of the semiclassical methods, starting from an equivalent way to derive the Fourier transform of the kink solution. To simplify the notation, let's get rid of all possible constants and consider the Fourier transform of the derivative of the kink solution, expressed as

$$G(k) = \int_{-\infty}^{\infty} dx e^{ikx} \frac{1}{\cosh^2 x} . \quad (3.26)$$

We split the integral in two terms

$$G(k) = \int_{-\infty}^0 dx e^{ikx} \frac{1}{\cosh^2 x} + \int_0^{\infty} dx e^{ikx} \frac{1}{\cosh^2 x} , \quad (3.27)$$

and we use the following series expansion of the integrand, valid on the entire real axis (except the origin)

$$\frac{1}{\cosh^2 x} = 4 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-2n|x|} . \quad (3.28)$$

Substituting this expression into (3.27) and computing each integral, we have

$$G(k) = 4i \sum_{n=1}^{\infty} (-1)^{n+1} n \left[-\frac{1}{ik + 2n} + \frac{1}{-ik + 2n} \right] . \quad (3.29)$$

Obviously it coincides with the exact result, $G(k) = \pi k / \sinh \frac{\pi}{2} k$, but this derivation permits to easily interpret the physical origin of each pole. In fact, changing k to the original variable in the crossed channel, $k \rightarrow (i\pi - \theta)/\xi$, we see that the poles

which determine the bound states at the vacuum $|a\rangle$ are only those relative to the exponential behaviour of the kink solution at $x \rightarrow -\infty$. This is precisely the point where the classical kink solution takes values on the vacuum $|a\rangle$. In the case of φ^4 , the kink and the antikink are the same function (up to a minus sign) and therefore they have the same exponential approach at $x = -\infty$ at both vacua $|\pm 1\rangle$. Mathematically speaking, this is the reason for the coincidence of the bound state spectrum on each of them: this does not necessarily happens in other cases, as we will see in the next section, for instance.

The second comment concerns the behavior of the kink solution near the minima of the potential. In the case of φ^4 , expressing the kink solution as

$$\varphi(x) = \frac{m}{\sqrt{\lambda}} \tanh \left[\frac{m x}{\sqrt{2}} \right] = \frac{m}{\sqrt{\lambda}} \frac{e^{\sqrt{2}x} - 1}{e^{\sqrt{2}x} + 1} , \quad (3.30)$$

and expanding around $x = -\infty$, we have

$$\varphi(t) = -\frac{m}{\sqrt{\lambda}} \left[1 - 2t + 2t^2 - 2t^3 + \dots 2(-1)^n t^n \dots \right] , \quad (3.31)$$

where $t = \exp[\sqrt{2}x]$. Hence, all the sub-leading terms are exponential factors, with exponents which are multiple of the first one. Is this a general feature of the kink solutions of any theory? It can be proved that the answer is indeed positive [27].

The fact that the approach to the minimum of the kink solutions is always through multiples of the same exponential (when the curvature ω at the minimum is different from zero) implies that the Fourier transform of the kink solution has poles regularly spaced by $\xi_a \equiv \frac{\omega}{\pi M_{ab}}$ in the variable θ . If the first of them is within the physical strip, the semiclassical mass spectrum derived from the formula (2.9) near the vacuum $|a\rangle$ has therefore the universal form

$$m_n = 2M_{ab} \sin \left(n \frac{\pi \xi_a}{2} \right) . \quad (3.32)$$

As we have previously discussed, this means that, according to the value of ξ_a , we can have only the following situations at the vacuum $|a\rangle$: (a) no bound state if $\xi_a > 1$; (b) one particle if $\frac{1}{2} < \xi_a < 1$; (c) two particles if $\frac{1}{3} < \xi_a < \frac{1}{2}$; (d) $\left\lceil \frac{1}{\xi_a} \right\rceil$ particles if $\xi_a < \frac{1}{3}$, although only the first two are stable, the others being resonances. So, semiclassically, each vacuum of the theory cannot have more than two stable particles above it. Viceversa, if $\omega = 0$, there are no poles in the Fourier transform of the kink and therefore there are no neutral particles near the vacuum $|a\rangle$.

4 Asymmetric wells

In order to have a polynomial potential with two asymmetric wells, one must necessarily employ higher powers than φ^4 . The simplest example of such a potential is obtained with a polynomial of maximum power φ^6 , and this is the example discussed here. Apart from its simplicity, the φ^6 theory is relevant for the class of universality of the Tricritical Ising Model [28]. As we can see, the information available on this model will turn out to be a nice confirmation of the semiclassical scenario. .

A class of potentials which may present two asymmetric wells is given by

$$U(\varphi) = \frac{\lambda}{2} \left(\varphi + a \frac{m}{\sqrt{\lambda}} \right)^2 \left(\varphi - b \frac{m}{\sqrt{\lambda}} \right)^2 \left(\varphi^2 + c \frac{m^2}{\lambda} \right) , \quad (4.33)$$

with a, b, c all positive numbers. To simplify the notation, it is convenient to use the dimensionless quantities obtained by rescaling the coordinate as $x^\mu \rightarrow mx^\mu$ and the field as $\varphi(x) \rightarrow \sqrt{\lambda}/m\varphi(x)$. In this way the lagrangian of the model becomes

$$\mathcal{L} = \frac{m^6}{\lambda^2} \left[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\varphi + a)^2(\varphi - b)^2(\varphi^2 + c) \right] . \quad (4.34)$$

The minima of this potential are localised at $\varphi_0^{(0)} = -a$ and $\varphi_1^{(0)} = b$ and the corresponding ground states will be denoted by $|0\rangle$ and $|1\rangle$. The curvature of the potential at these points is given by

$$\begin{aligned} U''(-a) &\equiv \omega_0^2 = (a+b)^2(a^2+c) ; \\ U''(b) &\equiv \omega_1^2 = (a+b)^2(b^2+c) . \end{aligned} \quad (4.35)$$

For $a \neq b$, we have two asymmetric wells, as shown in Figure 7. To be definite, let's assume that the curvature at the vacuum $|0\rangle$ is higher than the one at the vacuum $|1\rangle$, i.e. $a > b$.

The problem we would like to examine is whether the spectrum of the neutral particles $|B\rangle_s$ ($s = 0, 1$) may be different at the two vacua, in particular, whether it would be possible that one of them (say $|0\rangle$) has no neutral excitations, whereas the other has just one neutral particle. The ordinary perturbation theory shows that both vacua has neutral excitations, although with different value of their mass:

$$m^{(0)} = (a+b)\sqrt{2(a^2+c)} \quad , \quad m^{(1)} = (a+b)\sqrt{2(b^2+c)} . \quad (4.36)$$

Let's see, instead, what is the semiclassical scenario. The kink equation is given in this case by

$$\frac{d\varphi}{dx} = \pm(\varphi + a)(\varphi - b) \sqrt{\varphi^2 + c} . \quad (4.37)$$

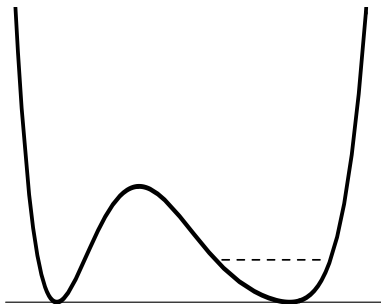


Figure 7: *Example of φ^6 potential with two asymmetric wells and a bound state only on one of them.*

We will not attempt to solve exactly this equation but we can present nevertheless its main features. The kink solution interpolates between the values $-a$ (at $x = -\infty$) and b (at $x = +\infty$). The anti-kink solution does viceversa, but with an important difference: its behaviour at $x = -\infty$ is different from the one of the kink. As a matter of fact, the behaviour at $x = -\infty$ of the kink is always equal to the behaviour at $x = +\infty$ of the anti-kink (and viceversa), but the two vacua are approached, in this theory, differently. This is explicitly shown in Figure 8 and proved in the following.

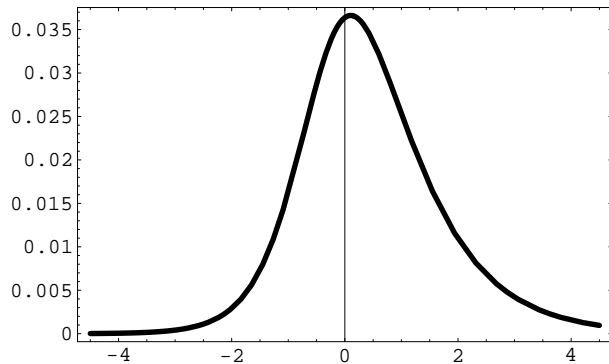


Figure 8: *Typical shape of $(\frac{d\varphi}{dx})_{0,1}$, obtained by a numerical solution of eq. (4.37).*

Let us consider the limit $x \rightarrow -\infty$ of the kink solution. For these large values of x , we can approximate eq. (4.37) by substituting, in the second and in the third term of the right-hand side, $\varphi \simeq -a$, with the result

$$\left(\frac{d\varphi}{dx}\right)_{0,1} \simeq (\varphi + a)(a + b)\sqrt{a^2 + c} \quad , \quad x \rightarrow -\infty \quad (4.38)$$

This gives rise to the following exponential approach to the vacuum $|0\rangle$

$$\varphi_{0,1}(x) \simeq -a + A \exp(\omega_0 x) \quad , \quad x \rightarrow -\infty \quad (4.39)$$

where $A > 0$ is a arbitrary costant (its actual value can be fixed by properly solving the non-linear differential equation). To extract the behavior at $x \rightarrow -\infty$ of the anti-kink, we substitute this time $\varphi \simeq b$ into the first and third term of the right hand side of (4.37), so that

$$\left(\frac{d\varphi}{dx}\right)_{1,0} \simeq (\varphi - b)(a + b)\sqrt{b^2 + c} \quad , \quad x \rightarrow -\infty \quad (4.40)$$

This ends up in the following exponential approach to the vacuum $|1\rangle$

$$\varphi_{1,0}(x) \simeq b - B \exp(\omega_1 x) \quad , \quad x \rightarrow -\infty \quad (4.41)$$

where $B > 0$ is another constant. Since $\omega_0 \neq \omega_1$, the asymptotic behaviour of the two solutions gives rise to the following poles in their Fourier transform

$$\begin{aligned} \mathcal{F}(\varphi_{0,1}) &\rightarrow \frac{A}{\omega_0 + ik} \\ \mathcal{F}(\varphi_{1,0}) &\rightarrow \frac{-B}{\omega_1 + ik} \end{aligned} \quad (4.42)$$

In order to locate the pole in θ , we shall reintroduce the correct units. Assuming to have solved the differential equation (4.37), the integral of its energy density gives the common mass of the kink and the anti-kink. In terms of the constants in front of the Lagrangian (4.34), its value is given by

$$M = \frac{m^5}{\lambda^2} \alpha \quad , \quad (4.43)$$

where α is a number (typically of order 1), coming from the integral of the adimensional energy density (1.4). Hence, the first pole² of the Fourier transform of the kink and the antikink solution are localised at

$$\begin{aligned} \theta^{(0)} &\simeq i\pi \left(1 - \omega_0 \frac{m}{\pi M}\right) = i\pi \left(1 - \omega_0 \frac{\lambda^2}{\alpha m^4}\right) \\ \theta^{(1)} &\simeq i\pi \left(1 - \omega_1 \frac{m}{\pi M}\right) = i\pi \left(1 - \omega_1 \frac{\lambda^2}{\alpha m^4}\right) \end{aligned} \quad (4.44)$$

²In order to determine the others, one should look for the subleading exponential terms of the solutions.

If we now choose the coupling constant in the range

$$\frac{1}{\omega_0} < \frac{\lambda^2}{m^4} < \frac{1}{\omega_1} , \quad (4.45)$$

the first pole will be out of the physical sheet whereas the second will still remain inside it! Hence, the theory will have only one neutral bound state, localised at the vacuum $|1\rangle$. This result may be expressed by saying that the appearance of a bound state depends on the order in which the topological excitations are arranged: an antikink-kink configuration gives rise to a bound state whereas a kink-antikink does not.

Finally, notice that the value of the adimensional coupling constant can be chosen so that the mass of the bound state around the vacuum $|1\rangle$ becomes equal to mass of the kink. This happens when

$$\frac{\lambda^2}{m^4} = \frac{\alpha}{3\omega_1} . \quad (4.46)$$

Strange as it may appear, the semiclassical scenario is well confirmed by an explicit example. This is provided by the exact scattering theory of the Tricritical Ising Model perturbed by its sub-leading magnetization. Firstly discovered through a numerical analysis of the spectrum of this model [29], its exact scattering theory has been discussed later in [30].

5 Conclusions

In this paper we have used simple arguments of the semi-classical analysis to investigate the spectrum of neutral particles in quantum field theories with kink excitations. We have concentrated our analysis on two cases: the first relative to a potential with symmetric wells, the second concerning with a potential with asymmetric wells. Leaving apart the exact values of the quantities extracted by the semiclassical methods, it is perhaps more important to underline some general features which have emerged through this analysis. One of them concerns, for instance, the existence of a critical value of the coupling constant, beyond which there are no neutral bound states. Another result is about the maximum number $n \leq 2$ of neutral particles living on a generic vacuum of a non-integrable theory. An additional aspect is the role played by the asymmetric vacua, which may have a different number of neutral excitations above them.

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References

- [1] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Ann. Phys.* **120** (1979) 253.
- [2] A.B. Zamolodchikov, *Adv. Stud. Pure Math.* **19** (1989), 641.
- [3] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, (World Scientific, Singapore, 1992); M. Karowski and P. Weisz, *Nucl. Phys.* **D 139**, (1978), 455.
- [4] G. Delfino and G. Mussardo, *Nucl. Phys.* **B 455**, (1995), 724; G. Delfino and P. Simonetti, *Phys. Lett.* **B 383**, (1996), 450.
- [5] G. Mussardo, *Phys. Rept.* **218** (1992), 215.
- [6] Al.B.Zamolodchikov, *Nucl.Phys.* **B 358**, (1991), 524.
- [7] A.B.Zamolodchikov and Al.B.Zamolodchikov, *Nucl.Phys.* **B 379**, (1992), 602.
- [8] P. Fendley, H. Saleur and N.P. Werner, *Nucl.Phys.* **B 430**, (1994), 577.
- [9] G.Delfino, G.Mussardo and P.Simonetti, *Phys. Rev.* **D 51**, (1995), 6622.
- [10] G.Delfino, G.Mussardo and P.Simonetti, *Nucl.Phys.* **B 473**, (1996), 469.
- [11] P. Grinza, G. Delfino and G. Mussardo, hep/th 0507133, *Nucl. Phys. B* in press.
- [12] G. Delfino, *Particle decay in Ising field theory with magnetic field*, Proceedings ICMP 2006.
- [13] B.M. McCoy and T.T. Wu, *Phys. Rev.* **D 18** (1978), 1259.
- [14] P. Fonseca and A.B. Zamolodchikov, *J.Stat.Phys.* **110** (2003), 527.
- [15] S.B. Rutkevich, *Phys. Rev. Lett.* **95** (2005), 250601.
- [16] P. Fonseca and A.B. Zamolodchikov, *Ising Spectroscopy I: Mesons at $T < T_c$* , hep-th/0612304.

- [17] G. Delfino and G. Mussardo, *Nucl. Phys.* **B 516**, (1998), 675.
- [18] Z. Bajnok, L. Palla, G. Takacs, F. Wagner, *Nucl.Phys.* **B 601**, (2001), 503.
- [19] D. Controzzi and G. Mussardo, *Phys. Rev. Lett.* **92**, (2004), 021601.
- [20] D. Controzzi and G. Mussardo, *Phys. Lett.* **B 617**, (2005), 133.
- [21] G. Delfino, P. Grinza and G. Mussardo, *Nucl. Phys.* **B 737** (2006), 291.
- [22] R.F.Dashen, B.Hasslacher and A.Neveu, *Phys. Rev.* **D 10** (1974) 4130;
R.F.Dashen, B.Hasslacher and A.Neveu, *Phys. Rev.* **D 11** (1975) 3424.
- [23] J. Goldstone and R. Jackiw, *Phys.Rev.* **D 11** (1975) 1486.
- [24] R. Jackiw and G. Woo, *Phys. Rev.* **D 12** (1975), 1643.
- [25] G. Mussardo, V. Riva and G. Sotkov, *Nucl. Phys.* **B 670** (2003), 464.
- [26] G. Mussardo, V. Riva and G. Sotkov, *Nucl. Phys.* **B 699** (2004), 545.
G. Mussardo, V. Riva and G. Sotkov, *Nucl. Phys.* **B 705** (2005), 548
- [27] G. Mussardo, *Neutral bound states in kink-like theories*, hep-th/0607025, to appear on Nucl. Phys. B.
- [28] A.B. Zamolodchikov, *Sov.J.Nucl.Phys.* **44** (1986), 529.
- [29] M. Lassig, G. Mussardo and J.L. Cardy, *Nucl. Phys.* **B 348** (1991), 591.
- [30] F. Colomo, A. Koubek and G. Mussardo, *Int. Journ. Mod. Phys.* **A 7** (1992), 5281.